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Exponentially-fitted Third-Order Adams-Bashfort Method

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Abstract— In this paper, the construction of Exponentially-Fitted (EF) versions of the third-order Adam-Bashfort method for oscillatory problems is presented. The convergence and stability properties of the constructed methods are investigated. Numerical experiments confirming the theoretical expectations regarding the constructed methods compared with other standard classical methods are also presented.

Keywords— Exponential–Fitting; Adams–Bashfort; scillatory problems; Ordinary Differential Equations

1. Introduction

Development of new methods for the numerical integration of initial value problems

$$u^{(n)} = f\left(t, u, u', u'', \cdots, u^{(n-1)}\right), t \in [t_0, T], u^{(v)}(t_0) = \eta_v, v = 0,1,2, \cdots, n-1 \quad (1)$$

Whose solution exhibits a pronounced oscillatory behavior is recently increasing. Several problems in electronics, dynamics and mechanics are often of this form. Efficient adaptation of existing classical methods for (1) is necessary as they are not well–suited for this type of problem [8]. Exponentially–fitted formulae were first proposed [9] for approaching the solution of stiff differential equations. A–stable fourth order exponentially–fitted formulae based on a linear 2-step formula was constructed in [7]. The author in [3], derived exponentially–fitted Multi derivative Linear Multistep Method (MLMM) involving the second derivative formulae. However, in the case of specially adapted methods, particular Runge–Kutta (RK) algorithms have been proposed by several authors [1, 2, 4, 5] in order to solve (1). One pioneer paper is the work due to [2], in which adapted RK algorithms with 3 and 4 stages for the integration of ODEs with oscillatory solutions are presented. The author in [1, 10] also constructed explicit RK methods which integrate certain first-order initial value problems with periodic or exponential solutions. On the other hand, exponentially fitted RK (EFRK) methods which integrate exactly first-order systems whose solutions can be expressed as linear combinations of functions of the form $\{exp(\lambda t), exp(-\lambda t)\}$ or $\{cos(\omega t), sin(\omega t)\}$ were introduced in [11, 12]. Exponentially–fitted 2-step Simpson's method has also be constructed by authors in [13]. In this paper, the construction of exponentially fitted Adams–Bashfort method based on the ideas proposed in [6, 11, 12] is analyzed.

2. Construction of the Exponentially Fitted Adams-Bashfort Methods

The classical third order Adams–Bashfort method for solving the first order IVP

$$u' = f(t, u), t \in [t_0, T], u(t_0) = u_0 \quad (2)$$

is given by

$$u_{n+1} = u_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}). \quad (3)$$

To derive the exponentially fitted versions of the classical Adams–Bashfort method, we rewrite (3) in a more general way as

$$u_{n+1} = a_0 u_n + h(b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2}). \quad (4)$$

Following the six-step procedure described in ([6]), the corresponding linear difference operator $L[h,a]$ reads

$$L[h,a]u(t) = u(t+h) - a_0 u(t) - h \quad (5)$$
$$where a := (a_0, b_0, b_1, b_2)$$

Applying step II of the six-step procedure we have the system

$$L_0\,(a) = 1 - a_0 = 0 \quad (6)$$
$$L_1\,(a) = 1 - b_0 - b_1 - b_2 = 0 \quad (7)$$
$$L_2\,(a) = 1 + 2b_1 + 4b_2 = 0 \quad (8)$$
$$L_3\,(a) = 1 - 3b_1 - 12b_2 = 0 \quad (9)$$

The algebraic system above is compatible and one finds $M = 4$.
Applying step III, we find that

$$G^{+(Z,a)= -a_0 + b_1\sqrt{Z}sinh(\sqrt{Z}) + b_2\sqrt{Z}sinh(2\sqrt{Z}) + cosh(\sqrt{Z})} \quad (10)$$

$$G^{-(Z,a)= -b_1 cosh(\sqrt{Z}) - b_2 cosh(2\sqrt{Z}) - b_0 + \frac{sinh(\sqrt{Z})}{\sqrt{Z}}} \quad (11)$$

where $z = \omega h = \omega_h$ and $Z = z^2$.
Implementing steps IV and V, we have only three choices:

- **<u>S1 :: (K,P) = (3,-1):</u>** By solving the system given by (6-9), we obtain

$$a_0 = 1, b_0 = \frac{23}{12}, b_1 = \frac{-4}{3}, b_2 = \frac{5}{12} \quad (12)$$

as coefficients for the classical Adams–Bashfort method (3).

- **<u>S2 :: (K,P) = (1,0):</u>** For this case, we have to solve the system given by (6-7), $G^{+(Z,a)=0}$ and $G^{-(Z,a)=0}$ and we find

$$a_0 = 1 \quad (13)$$

$$b_0 = \frac{-csch^2\left(\frac{\sqrt{Z}}{2}\right)\left(\sqrt{Z} + \left(sinh\left(\frac{3\sqrt{Z}}{2}\right) - sinh\left(\frac{5\sqrt{Z}}{2}\right)\right)sech\left(\frac{\sqrt{Z}}{2}\right)\right)}{4\sqrt{Z}} \quad (14)$$

$$b_1 = \frac{sinh(\sqrt{Z}) - sinh(2\sqrt{Z}) + \sqrt{Z}cosh(\sqrt{Z})}{\sqrt{Z}(cosh(\sqrt{Z}) - 1)} \quad (15)$$

$$b_2 = \frac{cosh(\sqrt{Z})csch(2\sqrt{Z})\left(2cosh(\sqrt{Z}) - \sqrt{Z}coth\left(\frac{\sqrt{Z}}{2}\right)\right)}{\sqrt{Z}} \quad (16)$$

- **<u>S3 :: (K,P) = (-1,1):</u>** For this case, we have to solve the system given by $G^{\pm}(Z,a) = 0$ and $G^{\pm'}(Z,a) = 0$ and we find

$$a_0 = \frac{-sinh(\sqrt{Z}) + sinh(3\sqrt{Z}) - \sqrt{Z}\left(cosh(3\sqrt{Z}) - 3cosh(\sqrt{Z})\right)}{2\sqrt{Z} + sinh(2\sqrt{Z})} \quad (17)$$

$$b_0 = \frac{csch(\sqrt{Z})\left(Z(4cosh(2\sqrt{Z}) + cosh(4\sqrt{Z}) - 3) - 2sinh^2(\sqrt{Z})\right)}{2Z\left(2\sqrt{Z} + sinh(2\sqrt{Z})\right)} \quad (18)$$

$$b_1 = \frac{2\sqrt{Z}sinh^2(\sqrt{Z}) + sinh(2\sqrt{Z}) - 2Zcosh(3\sqrt{Z})csch(\sqrt{Z})}{Z\left(2\sqrt{Z} + sinh(2\sqrt{Z})\right)} \quad (19)$$

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$$b_2 = \frac{Zcoth(2\sqrt{Z}) - \frac{tanh(\sqrt{Z})}{2}}{Z\left(sinh(\sqrt{Z}) + \sqrt{Z}sech(\sqrt{Z})\right)} \quad (20)$$

3. Error Analysis: Local Truncation Error (*lte*)

Following the six-step procedure [6], the general expression of the leading term of the local truncation error (*lte*) for an exponentially fitted method with respect to the basis functions

$$\{1, t, \cdots, t^K, exp(\pm\omega t), texp(\pm\omega t), \cdots, t^P exp(\pm\omega t)\} \quad (21)$$

takes the form

$$e^{EF}(t) = \quad (22)$$

with K, P and M satisfying the condition $K + 2P = M - 3$.
For the three methods constructed above, one finds the following results:

- **S1 ::** $(K, P) = (3, -1)$

$$e_{EF}(t) = \frac{3}{8}h^4 u^{(4)}(t) \quad (23)$$

- **S2 ::** $(K, P) = (1,0)$

$$e_{EF}(t) = -h^4 \frac{1}{2Z}\left(3 - \frac{2sinh\left(\frac{3\sqrt{Z}}{2}\right)sech\left(\frac{\sqrt{Z}}{2}\right)}{\sqrt{Z}}\right)\left(u^{(4)}(t) - \omega^2 u''(t)\right) \quad (24)$$

- **S3 ::** $(K, P) = (-1,1)$

$$e_{EF}(t) = h^4 \frac{4}{Z^2\left(2\sqrt{Z} + sinh(2\sqrt{Z})\right)}$$
$$\times \left(sinh^2\left(\frac{\sqrt{Z}}{2}\right)\left(-sinh(\sqrt{Z}) - sinh(2\sqrt{Z}) + 2\sqrt{Z}cosh(\sqrt{Z}) + \sqrt{Z}cosh(2\sqrt{Z})\right)\right)$$
$$\times \left(u^{(4)}(t) - 2\omega^2 u''(t) + \omega^4\right) \quad (25)$$

4. Convergence and Stability Analysis

Theorem 4.1 (Dahlquist Theorem) *The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable*

Dahlquist theorem (4.1) holds also true for EF-based algorithms but, because their coefficients are no longer constants the concepts of consistency and stability have to be adapted.

Definition 4.2 *An exponentially fitted method associated with the fitting space (21) is said to be of exponential order q, relative to the frequency ω if q is the maximum value of* ***M*** *such that the algebraic system* $\{L_m(a) = 0 \vee m = 0, \cdots, M - 1\}$ *can be solve.*

Definition 4.3 *A linear multistep method is said to be consistent if it has order* $p \geq 1$.

Since $M \geq 1$ for the scheme constructed aboves, the consistency requirement is satisfied. Hence, the derived schemes are all consistent. The stability regards the way how the errors accumulate when the solution is propagated along the interval of interest. The zero-stability refers to the limit case $h \to 0$ but in applications where only significantly non-vanishing steps are used. In [6] the first and second order equations were examined in detail in the exponential fitting context. The idea consists of choosing a differential equation whose analytic solution does not increase indefinitely when $x \to \infty$ and then checking whether the numerical solution conserves this property. For first order equations the test equation is $u' = \lambda u,\ t \geq 0$ with $R(\lambda) < 0$. Application of an s-step method on the test equation will lead to an s-order difference equation whose characteristic equation has s roots and the stability properties depend on the magnitude of these roots. For the versions presented above for the Adams–Bashforth methods with $\omega_h = \omega h$ and $Z = \omega_h^2$ the stability polynomial is given by

$$\pi(\xi, \acute{h}) = \rho(\xi) - \acute{h}\sigma(\xi)$$
$$\xi^3 - (a_0 + \acute{h}b_0)\xi^2 - \acute{h}b_1\xi - \acute{h}b_2 = 0 \tag{26}$$

where $\acute{h} = \lambda h$.

5. Numerical Results

Numerical experiments confirming the theoretical expectations regarding the constructed methods are now performed. The constructed methods are applied to two test problems and the result obtained compared with the classical third–order Adams–Bashforth, third–order Taylor's method and third–order explicit Runge-Kutta method.

5.1 Problem 1

Consider the IVP: $u' - u = t, u(0) = 1$ with the exact solution $u(t) = 2e^t - t - 1$.

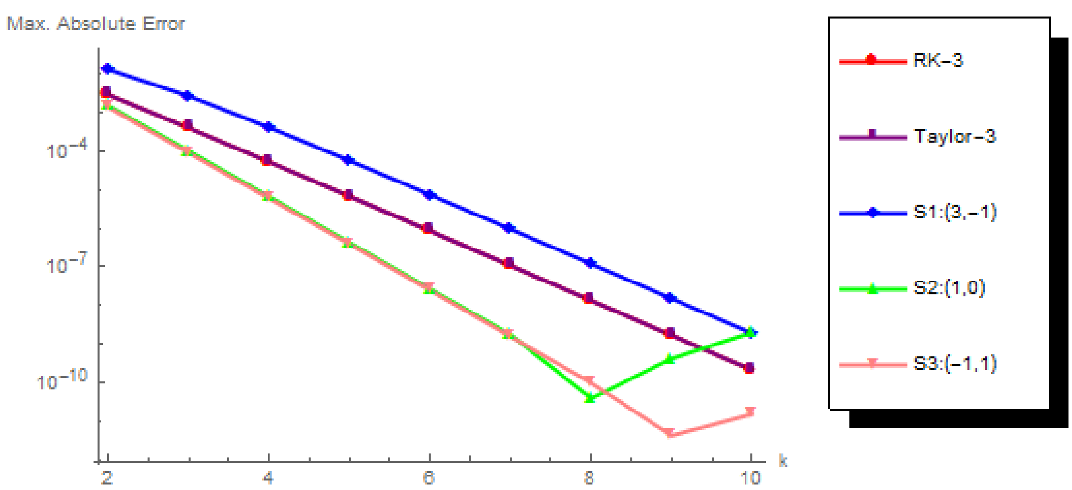

Figure 1: Maximum error as a function of the step-size $h = 2^{-k}, k = 2(1)10$

5.2 Problem 2

Consider the IVP: $u' = \alpha u + e^{\alpha t}, u(-1) = -e^{-\alpha}$ with the exact solution $u(t) = te^{\alpha t}$.

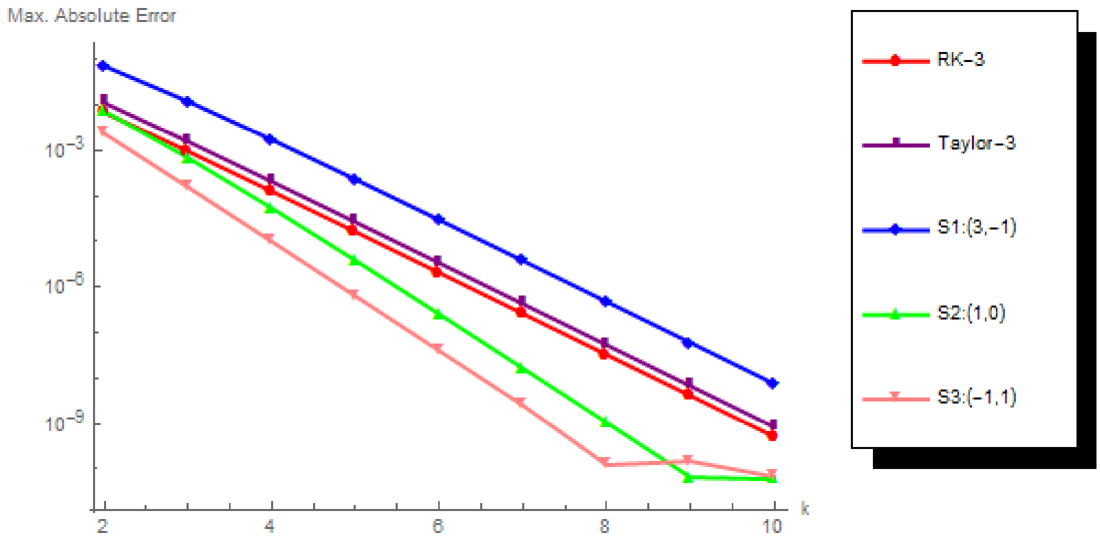

Figure 2: Maximum error as a function of the step-size $h = 2^{-k}, k = 2(1)10, \alpha = 1$

6. Conclusion

The exponentially–fitted versions of the Adams–Bashforth method have been constructed and implemented in this paper. The results obtained show that the theoretical expectations are meet. As expected, the exponentially fitted versions integrate the problems up to machine accuracy. It is also seen that the classical method performs poorly compared with the exponentially fitted counterparts.

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