British Journal of Computer Science and Information Technology

ISSN: 2227-9956
Volume 01, Issue 03, November, 2019

Common Fixed Point Theorems of Some Hybrid Iterative Schemes for Inequality Operators in Banach Spaces

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Abstract— In this paper, we introduce some hybrid iterative schemes (Picard-Ishikawa, Picard-AK, Picard-S hybrid iterative schemes) in line with Picard-Mann scheme and establish some strong convergence results for generalized contractive-like inequality operators introduced by Imoru and Olatinwo [10] in a Banach space. We also compare the convergence speed of CR, SP, Picard-Mann hybrid iterative schemes with our results (Picard-S, Picard-AK, Picard-Ishikawa hybrid iterative shemes). It is shown that one of our results (Picard-S iterative scheme) converges faster than others (CR, SP, Picard-AK, Picard-Ishikawa, Picard-Mann hybrid iterative schemes) for increasing functions, while the Picard-AK scheme is faster than CR scheme for decreasing function. Our results generalize and extend multitude of results in the literature, including the results of Khan [13].

Keywords— Strong convergence results, hybrid iterative schemes, rate of convergence, contractive-like operators.

1. Introduction and Preliminary Definitions

Fixed point iterative schemes are designed to be applied in solving equations arising in physical formulation but there is no systematic study of numerical aspects of these iterative schemes. In computational mathematics, it is of vital interest to know which of the given iterative procedures converge faster to a desired solution, commonly known as rate of convergence. We will now consider some of these schemes and compare their rate of convergence.

Let (X, d) be a metric space and $T: X \to X$ be a selfmap of X. Assume that $F_T = \{p \in X: T_p = p\}$ is the set of fixed points of T. For $u_0 \in X$, the sequence $\{u_n\}_{n=1}^{\infty}$ defined by

$$u_{n+1} = Tu_n, n \geq 0, \tag{1}$$

Is called the Picard iterative scheme.

We shall also need the following iterative schemes which appear in [15], [11], [16], [1], [19], [6], and [13] respectively to establish our results.

Let E be a Banach space and $T: E \to E$ a self-map of E. For $v_0 \in E$, the sequence $\{v_n\}_{n=0}^{\infty}$

$$v_{n+1} = (1-\alpha_n)v_n + \alpha_n Tv_n, n \geq 0, \tag{2}$$

Where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Mann iterative scheme [15].

If $\alpha_n = 1$ in (2), we have the Picard iterative scheme (1).
For $w_0 \in E$, the sequence $\{w_n\}_{n=0}^{\infty}$ defined by

$$w_{n+1} = (1-\alpha_n)w_n + \alpha_n Ty_n$$

$$y_n = (1-\beta_n)w_n + \beta_n T w_n, n \geq 0, \quad (3)$$

Where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0, 1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Ishikawa iterative scheme [11].
Observe that if $\beta_n = 0$ for each n, then the Ishikawa iterative scheme (3) reduces to the Mann iterative scheme (2).
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, n \geq 0, \end{aligned} \quad (4)$$

Where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0, 1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Noor iterative (or three-step) scheme [16].

Also observe that if $\gamma_n = 0$ for each n, then the Noor iteration process (3) reduces to the Ishikawa iterative scheme (2).

Rhoades [21], perhaps for the first time used computer programs to compare the rate of convergence Mann and Ishikawa iterative procedures. He illustrated the difference in the rate of convergence for increasing and decreasing functions through examples.
In 2007, Agarwal, Regan and Sahu introduced the following iterative scheme called S-iterative scheme [1] As: For $s_0 \in E$, the sequence $\{s_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} s_{n+1} &= (1-\alpha_n)Ts_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)s_n + \beta_n T s_n, n \geq 0, \end{aligned} \quad (5)$$

Where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0, 1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.
In [19], Phuengrattana and Suantai defined the SP iterative scheme and proved that this scheme is equivalent to and faster than Mann, Ishikawa and Noor iterative schemes for increasing functions. The Scheme is defined as follows: For $t_0 \in E$, $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} t_{n+1} &= (1-\alpha_n)y_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)t_n + \gamma_n T t_n, n \geq 0, \end{aligned} \quad (6)$$

Where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that

$\sum_{n=0}^{\infty} \alpha_n = \infty$. Recently, Chugh and Kumar [7] introduced the following CR iterative

scheme:

$$\begin{aligned} &e_0 \in E, \{x_n\}_{n=0}^{\infty} \\ &e_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \\ &y_n = (1-\beta_n)Te_n + \beta_n T z_n, \\ &z_n = (1-\gamma_n)e_n + \gamma_n T e_n, n \geq 0, \\ &\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \end{aligned}$$

Where are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. (7)
In 2013, Khan [13], introduced the following Picard-Mann hybrid iterative scheme for a single non expansive mapping T. For any initial point $a_0 \in E$ the sequence $\{a_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} a_{n+1} &= T y_n \\ y_n &= (1-\alpha_n)a_n + \alpha_n T a_n, n \geq 0, \end{aligned} \quad (8)$$

Where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1].
He showed that the new scheme (Picard-Mann scheme (8)) converges faster than all of Picard (1), Mann (2) And Ishikawa (3) iterative schemes in the sense of Berinde [5] for contractions. He also proved strong convergence and weak convergence theorems with the help of his iterative process (8) for the class of non-expansive mappings in general Banach spaces and apply it to obtain a result in uniformly convex Banach spaces.
Motivated by the work of Khan [13], we introduce the following hybrid iterative schemes and prove their

Strong convergence results for contractive-like operators [10] in Banach spaces. Also, we investigate their rate of convergence for this class of operators.

For any initial point $b_0 \in E$, the sequence $\{b_n\}_{n=0}^{\infty}$ is defined by:

$$\begin{aligned} b_{n+1} &= Ty_n, \\ y_n &= (1-\alpha_n)b_n + \alpha_n Tz_n, \\ z_n &= (1-\beta_n)b_n + \beta_n Tb_n, n \geq 0, \end{aligned} \tag{9}$$

Where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0, 1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Picard-Ishikawa hybrid iterative scheme.

For any initial point $c_0 \in E$, the sequence $\{c_n\}_{n=0}^{\infty}$ is defined by:

$$\begin{aligned} c_{n+1} &= Ty_n, \\ y_n &= (1-\alpha_n)z_n + \alpha_n Tz_n, \\ z_n &= (1-\beta_n)c_n + \beta_n Tc_n, n \geq 0, \end{aligned} \tag{10}$$

Where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0, 1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Picard-AK hybrid iterative scheme.

For any initial point $d_0 \in E$ the sequence is defined by

$$\begin{aligned} d_{n+1} &= Ty_n, \\ y_n &= (1-\alpha_n)Td_n + \alpha_n Tz_n, \\ z_n &= (1-\beta_n)d_n + \beta_n Td_n, n \geq 0, \end{aligned} \tag{11}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Picard-S hybrid Iterative scheme.

Several generalizations of the Banach fixed point theorem have been proved to date, (for example see [5], [6], [12], [13], [18] and [24]). One of the most commonly studied generalization hitherto is the one proved by Zamfirescu [26] in 1972, which is stated as thus:

Theorem 1.1: Let X be a complete metric space and $T: X \to X$ a Zamfirescu operator satisfying

$$d(Tx,Ty) \leq h\, max\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)], \frac{1}{2}[d(x,Ty) + d(y,Ty)], \tag{12}$$

where $0 \leq h < 1$. Then, T has a unique fixed point and the Picard iteration (1) converges to p for any $x_0 \in X$.

Observe that in a Banach space setting, condition (12) implies

$$\| Tx - Ty \| \leq \delta \| x - y \| + 2\delta \| x - Tx \|, \tag{13}$$

where $0 \leq \delta < 1$ and $\delta = \max\{h, \frac{h}{2-h}\}$, for details of proof (see[5]).

Several papers have been written on the Zamfirescu operators (13), for example (see [5], [8], [20], [26]). The most commonly used methods of approximating the fixed points of the Zamfirescu operators are Picard, Mann [15], Ishikawa [11] and Noor [16] iterative schemes. Rhoades [21, 22] used the Zamfirescu contraction condition (13) to obtain some convergence results for Mann and Ishikawa iterative schemes in a uniformly Banach space. Berinde [5] extended the results of the author [21, 22] to arbitrary Banach space for the same fixed point iteration procedures. Rafiq [20], proved the convergence of Noor iterative scheme using the Zamfirescu operators defined by (13).

Osilike [18] proved several stability results which are generalizations and extensions of most of the results of Rhoades [22] using the following contractive definition: for each $x, y \in X$, there exist $a \in [0,1)$

And $L \geq 0$ such that

$$d(Tx, Ty) \leq ad(x, y) + Ld(x, Tx). \tag{14}$$

In 2003, Imoru and Olatinwo [10] proved some stability results using the following general contractive definition:

For each $x, y \in X$, there exist $\delta \in [0,1)$ and a monotone increasing function $\varphi: R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + \varphi(d(x, Tx)). \tag{15}$$

2. Main Results

Theorem 2.1: Let K be a nonempty closed convex subset of an arbitrary Banach space $(E, ||.||)$ and be $T: K \to K$ a selfmap of K satisfying the condition

$$\| Tx - Ty \| \leq \delta \| x - y \| + \varphi(\| x - Tx \|), \tag{16}$$

for each $x, y \in K$, $0 \leq \delta < 1$ and $\varphi: R^+ \to R^+$ a monotone increasing function with $\varphi(0) = 0$. For $b_0 \in K$, let $\{b_n\}_{n=0}^{\infty}$ be the Picard-Ishikawa hybrid iterative scheme defined by (9), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$. Then

(i) T defined by (16) has a unique fixed point p.

(ii) the Picard-Ishikawa hybrid iterative scheme defined by (9) converges strongly to p of T.

Proof:

(i) We shall first establish that the mapping T satisfying the contractive condition (16) has a unique fixed point.

Suppose there exist $p_1, p_2 \in F_T$, and that $p_1 \neq p_2$, with $\| p_1 - p_2 \| > 0$, then

$$\begin{aligned} \| p_1 - p_2 \| = \| Tp_1 - Tp_2 \| &\leq \varphi(\| p_1 - Tp_1 \|) + \delta \| p_1 - p_2 \| \\ &= \varphi(0) + \delta \| p_1 - p_2 \|, \end{aligned} \tag{17}$$

Thus,

$$(1 - \delta) \| p_1 - p_2 \| \leq 0. \tag{18}$$

Since $\delta \in [0,1)$, then $1 - \delta > 0$ and $\| p_1 - p_2 \| \leq 0$. Also, since norm is nonnegative we have that $\| p_1 - p_2 \| = 0$. That is, $p_1 = p_2 = p$ (say). Thus, T has a unique fixed point p.

Next, we will establish that $lim_{n\to\infty} b_n = p$. That is, we show that the Picard-Ishikawa hybrid iterative scheme (9) converges strongly to p of T

Proof:

In view of (16) and (9), we have

$$\begin{aligned} \| b_{n+1} - p \| &= \| Ty_n - Tp \| \\ &\leq \delta \| y_n - p \| + \varphi(\| p - Tp \|) \\ &= \delta \| y_n - p \|. \end{aligned} \tag{19}$$

$$\begin{aligned} \| y_n - p \| &\leq (1 - \alpha_n) \| b_n - p \| + \alpha_n \| Tz_n - Tp \| \\ &\leq (1 - \alpha_n) \| b_n - p \| + \alpha_n [\delta \| z_n - p \| + \varphi(\| p - Tp \|)] \\ &= (1 - \alpha_n) \| b_n - p \| + \delta \alpha_n \| z_n - p \|. \end{aligned} \tag{20}$$

$$\begin{aligned} \| z_n - p \| &\leq (1 - \beta_n) \| b_n - p \| + \beta_n \| Tb_n - Tp \| \\ &\leq (1 - \beta_n) \| b_n - p \| + \beta_n [\delta \| b_n - p \| + \varphi(\| p - Tp \|)] \\ &= (1 - \beta_n) \| b_n - p \| + \delta \beta_n \| b_n - p \|. \end{aligned} \tag{21}$$

Substituting (21) in (18), we have

$$\| y_n - p \| \le (1-\alpha_n) \| b_n - p \| + \delta\alpha_n[(1-\beta_n) \| b_n - p \| + \delta\beta_n \| b_n - p \|]. \qquad (22)$$

Substituting (22) in (17), we have

$$\| b_{n+1} - p \| \le \delta[(1-\alpha_n) \| b_n - p \| + \delta\alpha_n[(1-\beta_n) \| b_n - p \| + \delta\beta_n \| b_n - p \|]]$$
$$= \delta[1-(1-\delta)\alpha_n - (1-\delta)\delta\alpha_n\beta_n] \| b_n - p \|. \qquad (23)$$

Using the fact that $0 \le \delta < 1$, $\alpha_n, \beta_n \in [0,1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, it result that $\lim_{n\to\infty} \| b_{n+1} - p \| = 0$. That is $\{b_n\}_{n=0}^{\infty}$ converges strongly to P. This ends the proof.

Remark 2.2: Theorem 2.1 generalizes several results in literature by considering a larger class of inequality operators (or contractive-like operators) (16).

Theorem 2.1. Leads to the following corollary:

Corollary 2.3: Let K be a nonempty closed convex subset of an arbitrary Banach space $(E, ||.||)$ and $T: K \to K$ be a self-map of K satisfying the condition

$$\| Tx - Ty \| \le \delta \| x - y \| + \varphi(\| x - Tx \|), \qquad (24)$$

for each $x, y \in K$, $0 \le \delta < 1$ and $\varphi: R^+ \to R^+$ a monotone increasing function with $\varphi(0) = 0$. For $a_0, u_0 \in K$, let $\{a_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be the Picard-Mann hybrid and Picard iterative schemes defined by (8) and (1) respectively, where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$. Then

(i) T defined by (24) has a unique fixed point p;
(ii) the Picard-Mann hybrid (10) iterative scheme converges strongly to p;
(iii) the Picard (1) iterative scheme converges strongly to p.

Theorem 2.4: Let K be a nonempty closed convex subset of an arbitrary Banach space $(E, ||.||)$ and be $T: K \to K$ a selfmap of K satisfying the condition

$$\| Tx - Ty \| \le \delta \| x - y \| + \varphi(\| x - Tx \|), \qquad (25)$$

for each $x, y \in K$, $0 \le \delta < 1$ and $\varphi: R^+ \to R^+$ a monotone increasing function with $\varphi(0) = 0$. For $c_0 \in K$, let $\{c_n\}_{n=0}^{\infty}$ be the Picard-AK hybrid iterative scheme defined by (??), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ Are real sequences in $[0,1]$. then

(i) T defined by (25) has a unique fixed point p.
(ii) The Picard-AK hybrid iterative scheme (10) converges strongly to p of T.

Proof:
(i) Trivial (the method of proof is similar to that in theorem 2.1).
(ii) Next, we shall establish that $lim_{n\to\infty} c_n = p$. That is, we show that the Picard-AK hybrid iterative scheme (10) converges strongly to p of T

Proof:
In view of (25) and (10), we have

$$\| c_{n+1} - p \| = \| Ty_n - Tp \|$$
$$\le \delta \| y_n - p \| + \varphi(\| p - Tp \|)$$
$$= \delta \| y_n - p \|. \qquad (26)$$

$$\| y_n - p \| \le (1-\alpha_n) \| z_n - p \| + \alpha_n \| Tz_n - Tp \|$$
$$\le (1-\alpha_n) \| z_n - p \| + \alpha_n[\delta \| z_n - p \| + \varphi(\| p - Tp \|)]$$
$$= (1-\alpha_n) \| z_n - p \| + \delta\alpha_n \| z_n - p \|$$
$$= (1-\alpha_n + \delta\alpha_n) \| z_n - p \|. \qquad (27)$$

$$\| z_n - p \| \le (1-\beta_n) \| c_n - p \| + \beta_n \| Tc_n - Tp \|$$
$$\le (1-\beta_n) \| c_n - p \| + \beta_n[\delta \| c_n - p \| + \varphi(\| p - Tp \|)]$$

$$= (1-\beta_n) \parallel c_n - p \parallel + \delta\beta_n \parallel c_n - p \parallel$$
$$= (1-\beta_n + \delta\beta_n) \parallel c_n - p \parallel. \quad (28)$$

Substituting (28) in (27), we have

$$\parallel y_n - p \parallel \leq (1-\alpha_n + \delta\alpha_n)(1-\beta_n + \delta\beta_n) \parallel c_n - p \parallel$$
$$= (1-\alpha_n + \delta\alpha_n)(1-\beta_n + \delta\beta_n) \parallel c_n - p \parallel. \quad (29)$$

Substituting (27) in (26), we have

$$\parallel c_{n+1} - p \parallel \leq \delta[(1-\alpha_n + \delta\alpha_n)(1-\beta_n + \delta\beta_n)] \parallel c_n - p \parallel$$
$$= \delta[(1-(1-\delta)\alpha_n)(1-(1-\delta)\beta_n)] \parallel c_n - p \parallel. \quad (30)$$

Using the fact that $0 \leq \delta < 1$, $\alpha_n, \beta_n \in [0,1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, it result that $\lim_{n\to\infty} \parallel c_{n+1} - p \parallel = 0$. That is, $\{c_n\}_{n=0}^{\infty}$ converges strongly to p. This ends the proof.

Theorem 2.4 leads to the following corollary:

Corollary 2.5. Let $(E, ||.||)$ be a normed linear space, $T: E \to E$ be a self-map of E satisfying the condition

$$\parallel Tx - Ty \parallel \leq \delta \parallel x - y \parallel + \varphi(\parallel x - Tx \parallel), \quad (31)$$

For each $x, y \in E$, $0 \leq \delta < 1$ and $\varphi: R^+ \to R^+$ a monotone increasing function with $\varphi(0) = 0$. For $a_0, u_0 \in K$, let $\{a_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be the Picard-Mann hybrid and Picard iterative schemes defined by (8) and (1) respectively, where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$. Then

(i) T defined by (31) has a unique fixed point p;

(ii) the Picard-Mann hybrid iterative scheme (8) converges strongly to p.

(iii)the Picard iterative scheme defined by (1) converges strongly to p.

Theorem 2.6: Let K be a nonempty closed convex subset of an arbitrary Banach space $(E, \parallel . \parallel)$ and $T: K \to K$ be a selfmap of K satisfying the condition

$$\parallel Tx - Ty \parallel \leq \delta \parallel x - y \parallel + \varphi(\parallel x - Tx \parallel), \quad (32)$$

for each $x, y \in K$, $0 \leq \delta < 1$ and $\varphi: R^+ \to R^+$ a monotone increasing function with $\varphi(0) = 0$. For $d_0 \in K$, let $\{d_n\}_{n=0}^{\infty}$ be the Picard-S hybrid iterative scheme defined by (11), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$. Then

(i) T defined by (32) has a unique fixed point p.

(ii) The Picard-S hybrid iterative scheme (11) converges strongly to p of T.

Proof:

(i) Trivial (the method of proof is similar to that in theorem (2.1)).

(ii) Next, we shall establish that $lim_{n\to\infty} d_n = p$. That is, we show that the Picard-S hybrid iterative scheme (11) converges strongly to p of T

Proof:

In view of (32) and (11), we have

$$\parallel d_{n+1} - p \parallel = \parallel Ty_n - Tp \parallel$$
$$\leq \delta \parallel y_n - p \parallel + \varphi(\parallel p - Tp \parallel)$$
$$= \delta \parallel y_n - p \parallel. \quad (33)$$

$$\parallel y_n - p \parallel \leq (1-\alpha_n) \parallel Td_n - p \parallel + \alpha_n \parallel Tz_n - Tp \parallel$$
$$\leq (1-\alpha_n)[\delta \parallel d_n - p \parallel + \varphi(\parallel p - Tp \parallel)]$$
$$+ \alpha_n[\delta \parallel z_n - p \parallel + \varphi(\parallel p - Tp \parallel)]$$
$$= (1-\alpha_n)\delta \parallel d_n - p \parallel + \delta\alpha_n \parallel z_n - p \parallel. \quad (34)$$

$$\parallel z_n - p \parallel \leq (1-\beta_n) \parallel d_n - p \parallel + \beta_n \parallel Td_n - Tp \parallel$$
$$\leq (1-\beta_n) \parallel d_n - p \parallel + \beta_n[\delta \parallel d_n - p \parallel + \varphi(\parallel p - Tp \parallel)]$$

$$= (1-\beta_n) \parallel d_n - p \parallel + \delta\beta_n \parallel d_n - p \parallel$$
$$= (1-\beta_n + \delta\beta_n) \parallel d_n - p \parallel. \quad (35)$$

Substituting (35) in (34), we have

$$\parallel y_n - p \parallel \leq (1-\alpha_n)\delta \parallel d_n - p \parallel + \delta\alpha_n(1-\beta_n+\delta\beta_n) \parallel d_n - p \parallel$$
$$= \delta[1-(1-\delta)\alpha_n\beta_n] \parallel d_n - p \parallel. \quad (36)$$

Substituting (36) in (33), we have

$$\parallel d_{n+1} - p \parallel \leq \delta^2[1-(1-\delta)\alpha_n\beta_n] \parallel d_n - p \parallel. \quad (37)$$

Using the fact that $0 \leq \delta < 1$, $\alpha_n, \beta_n \in [0,1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, it result that $\lim_{n\to\infty} \parallel d_{n+1} - p \parallel = 0$. That is, $\{d_n\}_{n=0}^{\infty}$ converges strongly to p. This ends the proof.

Following our method of proof, we now state the estimates of SP, CR, Noor, Ishikawa, Picard-Mann hybrid, Mann and Picard iterative schemes.

SP-iterative scheme:

$$\parallel t_{n+1} - p \parallel \leq [(1-(1-\delta)\alpha_n)(1-(1-\delta)\beta_n(1-(1-\delta)\gamma_n] \parallel t_n - p \parallel. \quad (38)$$

CR-iterative scheme:

$$\parallel e_{n+1} - p \parallel \leq \delta[(1-(1-\delta)\alpha_n - (1-\delta)(1-\alpha_n)\beta_n\gamma_n$$
$$-(1-\delta)(1-\alpha_n)\alpha_n\beta_n\gamma_n] \parallel e_n - p \parallel. \quad (39)$$

Noor iterative scheme:

$$\parallel x_{n+1} - p \parallel \leq [(1-(1-\delta)\alpha_n) - (1-\delta)\delta\alpha_n\beta_n$$
$$-(1-\delta)\delta^2\alpha_n\beta_n\gamma_n] \parallel x_n - p \parallel. \quad (40)$$

S- Iterative scheme:

$$\parallel s_{n+1} - p \parallel \leq \delta[1-(1-\delta)\alpha_n\beta_n] \parallel s_n - p \parallel. \quad (41)$$

Ishikawa- iterative scheme:

$$\parallel w_{n+1} - p \parallel \leq [1-(1-\delta)\alpha_n - (1-\delta)\alpha_n\beta_n] \parallel w_n - p \parallel. \quad (42)$$

Picard-Mann hybrid iterative scheme:

$$\parallel a_{n+1} - p \parallel \leq \delta[1-(1-\delta)\alpha_n] \parallel a_n - p \parallel. \quad (43)$$

Mann iterative scheme:

$$\parallel v_{n+1} - p \parallel \leq [1-(1-\delta)\alpha_n] \parallel v_n - p \parallel. \quad (44)$$

Picard iterative scheme:

$$\parallel u_{n+1} - p \parallel \leq \delta \parallel u_n - p \parallel. \quad (45)$$

Remark 2.7.

1. It already shown in [8] that CR iterative scheme converges faster than SP, S, Picard, Noor, Ishikawa and Mann Iterative schemes for increasing functions.
2. SP iterative scheme converges faster than CR, Mann, Noor, Ishikawa iterative schemes for decreasing functions. Picard and S iterations does not converge for decreasing functions.
3. We need to show if any of our new schemes perform better than the existing ones (stated in (1) and (2) above) for increasing or decreasing functions.

3. Numerical Example

In this section, we use known examples to compare our new iterative schemes (with higher precision, that

Is 12 decimal places) with others (CR, SP, Picard-Mann and Picard iterative schemes) with the help computer programs in MATHEMATICA 10.2. The results are shown in Tables 1-3, by taking initial approximation $d_0 = c_0 = b_0 = a_0 = e_0 = t_0 = 0.8$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{\frac{1}{2}}}$, for all the iterative schemes.

3.1. Example of Increasing Function

Let $f: [0,8] \rightarrow [0,8]$ be defined by $f(x) = \frac{x^2+9}{10}$. Then f is an increasing function. The comparison of These iterative schemes to the fixed point $p = 1$ is shown in Table 1.

3.2. Example of Decreasing Function

Let $f: [0,1] \rightarrow [0,1]$ be defined by $f(x) = (1-x)^m$, $m = 7,8, \ldots$. Then f is a decreasing function. By taking $m = 8$, the comparison of these iterative schemes to the fixed point p = 0.188347679972 is shown in Table 2.

3.3. Example of Cubic Equation

To find the root of the equation $x^3 + x^2 - 1 = 0$ means to find the fixed point of the function $(1-x^3)^{\frac{1}{2}}$ as $x^3 + x^2 - 1 = 0$ can be rewritten as $(1-x^3)^{\frac{1}{2}} = x$. The comparison of convergence of these various iterative schemes to the exact fixed point p = 0.754877666247 of $(1-x^3)^{\frac{1}{2}}$ is shown in Table 3.

Table 1: Numerical Example for Increasing Functions

n	Picard-S	Picard-AK	Picard-Ishikawa	Picard-Mann	CR	SP
	0.800000000000	0.800000000000	0.800000000000	0.800000000000	0.800000000000	0.800000000000
	0.994724335956	0.989107653287	0.981712510456	0.980046204397	0.985654249264	0.970029603170
	0.999845672115	0.999371298582	0.998236300270	0.997868571194	0.998875289673	0.995357217378
	0.999995473413	0.999963596112	0.999829084091	0.999770791148	0.999911279353	0.999277197126
	0.999999867220	0.999997891708	0.999983429280	0.999975333764	0.999992998079	0.999887385271
	0.999999996105	0.999999877899	0.999998393356	0.999997345347	0.999999447381	0.999982452208
	0.999999999886	0.999999992929	0.999999844224	0.999999714296	0.999999956385	0.999997265626
	0.999999999997	0.999999999590	0.999999984896	0.999999969251	0.999999996558	0.999999573917
	1.000000000000	0.999999999976	0.999999998536	0.999999996691	0.999999999728	0.999999933606
	-	0.999999999999	0.999999999854	0.999999999644	0.999999999979	0.999999989654
	-	1.000000000000	0.999999999986	0.999999999962	0.999999999998	0.999999998388
	-	-	0.999999999999	0.999999999996	1.000000000000	0.999999999749
	-	-	1.000000000000	1.000000000000	-	0.999999999961
	-	-	-	-	-	0.999999999994
	-	-	-	-	-	0.999999999999
	-	-	-	-	-	1.000000000000

Table 2: Numerical Example for Decreasing Functions

n	Picard-S	Picard-AK	Picard-Ishikawa	Picard-Mann	CR	SP
	0.800000000000	0.800000000000	0.800000000000	0.800000000000	0.800000000000	0.800000000000
	0.991661618055	0.038079720996	0.002903499306	0.002954193480	0.351281588128	0.230255307158
	0.999220874028	0.154923342340	0.896804076419	0.032981993771	0.253839245926	0.188347619027
	0.999301770329	0.188343056611	0.000966790483	0.063367134592	0.203435573050	0.188347679972
	0.999302594693	0.188347680815	0.910211859607	0.099708260081	0.189400928181	-

	0.999302603090	0.188347679972	0.000819059260	0.141220383659	0.188364553849	-
	0.999302603175	-	0.911237009301	0.174799938713	0.188347880562	-
	0.999302603176	-	0.000808619870	0.187104963671	0.188347682343	-
	0.999302603176	-	0.911309463821	0.188316696412	0.188347680000	-
	0.999302603176	-	0.000807886445	0.188347105317	0,188347679973	-
	0.999302603176	-	0.911314554213	0.188347669406	0.188347679972	-
	0.999302603176	-	.	0.188347679778	-	-
	0.999302603176	-	.	0.188347679969	-	-
	0.999302603176	-	.	0.188347679972	-	-

Table 3: Numerical Example for Cubic Equations

n	Picard-S	Picard-AK	Picard-Ishikawa	Picard-Mann	CR	SP
	0.800000000000	0.800000000000	0.800000000000	0.800000000000	0.800000000000	0.800000000000
	0.787843504237	0.754890266965	0.724951661134	0.755147653065	0.752481080032	0.754877149763
	0.778999243130	0.754877635539	0.771764753274	0.754863567955	0.754947140157	0.754877666195
	0.772549548615	0.754877666322	0.744382392790	0.754878407184	0.754875570498	0.754877666247
	0.767836695872	0.754877666247	0.761032699719	0.754877627320	0.754877729395	-
	0.764387522208	-	0.751140406670	0.754877668292	0.754877664344	-
	0.761860128333	-	0.757100088396	0.754877666139	0.754877666304	-
	0.760006476046	-	0.753539466597	0.754877666252	0.754877666245	-
	0.758646031672	-	0.755677436606	0.754877666246	0.754877666247	-
	0.757647057531	-	0.754397538130	0.754877666247	-	-
.	.	.	.	.	.	.
	0.754877671270	-	0.754877666247	-	-	-
.	.	.	.	.	.	.
	0.754877666250	-	-	-	-	-
	0.754877666249	-	-	-	-	-
	0.754877666248*	-	-	-	-	-

4. Observations

4.1. Increasing Function $f(x) = \frac{x^2+9}{10}$

The Picard-S hybrid iterative scheme converges to a fixed point in 8 iterations, Picard-AK hybrid scheme converges in 10 iterations, Picard-Ishikawa hybrid scheme converges in 12 iterations, Picard-Mann hybrid scheme converges in 12 iterations, CR scheme converges in 11 iterations and SP scheme converges in 15 iterations.

4.2. Decreasing Function $f(x) = (1-x)^m, m = 8.$

The Picard-S hybrid scheme shows a strange constant behaviour, Picard-AK hybrid scheme converges in 5 iterations, Picard-Ishikawa hybrid scheme oscillates between 0 and 1 (it never converges) for as many iterations, Picard-Mann hybrid scheme converges in 13 iterations, CR scheme converges in 10 iterations and SP scheme converges in 3 iterations.

4.3. Cubic Equation $x^3 + x^2 - 1 = 0.$

The Picard-S scheme converges to a different fixed point in 78 iterations, Picard-AK scheme converges in 4 iterations, Picard-Ishikawa scheme converges in 52 iterations, Picard-Mann scheme converges in 9 iterations, CR scheme converges in 8 iterations and SP scheme converges in 3 iterations.

4.4. Remarks

1. The order of decreasing rate of convergence in the case of increasing functions are: Picard-S, Picard-AK, Picard-Ishikawa, Picard-Mann, CR, SP iterative schemes.
2. The order of decreasing rate of convergence in the case of decreasing functions are: SP, Picard-AK,

CR, Picard-Mann iterative schemes.
3. The Picard-Ishikawa scheme does not converge for decreasing functions.
4. The order of decreasing rate of convergence in the case of cubic equation with m=8 are: SP, Picard-AK, CR, Picard- Mann, Picard- Ishikawa, Picard-S iterative schemes.
5. The number of steps increases as the precision is increased.

5. Conclusions

1. Our Picard-S hybrid scheme is faster than the others (Picard-AK, CR, Picard-Ishikawa, Picard-Mann, SP iterative schemes) for increasing functions.
2. SP scheme converges faster than Picard-AK, Picard-Mann and CR iterative schemes for decreasing functions.
3. Picard-AK hybrid scheme converges faster than Picard-Mann hybrid scheme for decreasing functions.
4. Picard-AK hybrid scheme converges faster than Picard-Mann and Picard-Ishikawa hybrid iterative schemes for the cubic equation considered.
5. SP scheme converges faster than Picard-AK, Picard-Mann, CR schemes and Picard-Ishikawa schemes for the same cubic equation.
6. The rate of convergence of these iterative schemes depend on the choice of $\alpha_n, \beta_n, \gamma_n$ and x_n from Tables 1-3.

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ISSN: 2227-9956
Volume 01, Issue 03, November, 2019

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